MATH 4030 Differential Geometry
Tutorial 7, 1 November 2017

1. (Expressing the 2nd fundamental form as Hessian) Let S be a surface in $\mathbb{R}^3$.

   (a) $\boxed{\text{height function}}$

   Let $\omega \in \mathbb{R}^3$ be a unit vector. Consider the function $f : S \to \mathbb{R} : x \mapsto \langle x, \omega \rangle$. By HW3 Q2(b), f is smooth and its set $\mathrm{Crit}(f)$ of critical points is precisely those $p \in S$ such that $T_pS \perp \omega$. Fix $p \in \mathrm{Crit}(f)$. Then $\mathrm{Hess}(f)_p$ is well-defined and we have

   $$\mathrm{Hess}(f)_p = (A_N)_p$$

   where the unit normal N to taken to be ω (since $\omega \perp T_pS$ for $p \in \mathrm{Crit}(f)$).

   Proof. Let $X : U \to V \subseteq S$ be a chart such that $X(\mathbf{0}) = p$. We have $f \circ X = \langle X, \omega \rangle$, $(f \circ X)_\diamondsuit = \langle X_\diamondsuit, \omega \rangle$ and $(f \circ X)_{\diamondsuit\heartsuit} = \langle X_{\diamondsuit\heartsuit}, \omega \rangle$ for any $\diamondsuit, \heartsuit = u$ or v. It follows that the matrix representing the Hessian of f at p with respect to $\{X_u, X_v\}$ is given by

   $$\mathrm{Hess}(f \circ X)_{\mathbf{0}} = \begin{pmatrix} \langle X_{uu}(\mathbf{0}), \omega \rangle & \langle X_{uv}(\mathbf{0}), \omega \rangle \\ \langle X_{uv}(\mathbf{0}), \omega \rangle & \langle X_{vv}(\mathbf{0}), \omega \rangle \end{pmatrix}$$

   which is the same as the matrix representing the 2nd fundamental form A_N of S at p with respect to $\{X_u, X_v\}$.

   Remark. If S is a graphical surface $\{(x, y, h(x, y)) |\ (x, y) \in U \subseteq \mathbb{R}^2\}$ and $dh(\mathbf{0}) = 0$, and if we take $\omega = (0, 0, 1)$, $p = (0, 0, h(0, 0))$, and $X : (u, v) \mapsto (u, v, h(u, v))$, then we have $f \circ X = h$ and we get the result in Lecture notes (Part 4) p.20-23.

   (b) $\boxed{\text{distance-square function}}$

   Let p_0 be a point not contained in S. Consider the function $g : S \to \mathbb{R} : x \mapsto |x - p_0|^2$. Then we can similarly show that g is smooth and $\mathrm{Crit}(g) = \{p \in S |\ T_pS \perp (p - p_0)\}$. Fix $p \in \mathrm{Crit}(g)$. Then we have

   $$\mathrm{Hess}(g)_p(\zeta, \theta) = 2[\langle \zeta, \theta \rangle_{\mathbb{R}^3} - |p_0 - p|(A_N)_p(\zeta, \theta)]$$

   for any $\zeta, \theta \in T_pS$, where N is taken to be $\frac{p_0 - p}{|p_0 - p|}$.

   Proof. Let $X : U \to V \subseteq S$ be a chart such that $X(\mathbf{0}) = p$. We have

   $$g \circ X = \langle X - p_0, X - p_0 \rangle;$$
   $$(g \circ X)_\diamondsuit(\mathbf{0}) = 2\langle X_\diamondsuit(\mathbf{0}), p - p_0 \rangle;$$
   $$(g \circ X)_{\diamondsuit\heartsuit}(\mathbf{0}) = 2\langle X_{\diamondsuit\heartsuit}(\mathbf{0}), p - p_0 \rangle + 2\langle X_\diamondsuit(\mathbf{0}), X_\heartsuit(\mathbf{0}) \rangle$$

   for any $\diamondsuit, \heartsuit = u$ or v. It follows that

   $$\mathrm{Hess}(g \circ X)_{\mathbf{0}} = 2\begin{pmatrix} \langle X_{uu}(\mathbf{0}), p - p_0 \rangle & \langle X_{uv}(\mathbf{0}), p - p_0 \rangle \\ \langle X_{uv}(\mathbf{0}), p - p_0 \rangle & \langle X_{vv}(\mathbf{0}), p - p_0 \rangle \end{pmatrix} + 2\begin{pmatrix} \langle X_u(\mathbf{0}), X_u(\mathbf{0}) \rangle & \langle X_u(\mathbf{0}), X_v(\mathbf{0}) \rangle \\ \langle X_u(\mathbf{0}), X_v(\mathbf{0}) \rangle & \langle X_v(\mathbf{0}), X_v(\mathbf{0}) \rangle \end{pmatrix}$$

   which is the same as the matrix representing the bilinear form

   $$(\zeta, \theta) \mapsto 2[\langle \zeta, \theta \rangle_{\mathbb{R}^3} - |p_0 - p|(A_N)_p(\zeta, \theta)]$$

   with respect to $\{X_u, X_v\}$.

2. See the upcoming Sol to HW4 Q7.